



TITLE:

Surjectivity of a Gluing for Stable T2-cones in Special Lagrangian Geometry(Digest_要約)

AUTHOR(S):

Imagi, Yohsuke

CITATION:

Imagi, Yohsuke. Surjectivity of a Gluing for Stable T2-cones in Special Lagrangian Geometry. 京都大学, 2014, 博士(理学)

ISSUE DATE:

2014-05-23

URL:

<https://doi.org/10.14989/doctor.k18444>

RIGHT:

学位規則第9条第2項により要約公開; 許諾条件により本文は2016-04-12に公開

Summary of ‘Surjectivity of a Gluing for Stable T^2 -cones in Special Lagrangian Geometry’

Yohsuke Imagi

Gluing is a method of constructing blowing-up solutions to partial differential equations, and we shall say that the gluing is surjective if all blowing-up solutions may be re-constructed by the gluing. In Yang–Mills gauge theory, for example, Taubes [19] constructs some instantons by the method of gluing, and Donaldson [4] proves its surjectivity; for the details we refer also to Freed and Uhlenbeck [6]. In special Lagrangian geometry, by a theory of Joyce [10, 11, 12, 13, 14], we can smooth some isolated conical singularities by the method of gluing, and we wish to prove its surjectivity.

We can formulate the surjectivity problems in terms of moduli spaces. In the situation of Donaldson’s result [4] indeed one has the moduli space of Yang–Mills instantons modulo gauge equivalence; using a result of Uhlenbeck [20] one can compactify it by adding some singular instantons; one defines the boundary of the moduli space as the set of all the singular instantons added for the compactification; using the surjectivity result one can determine a neighbourhood of the boundary in the compactified space.

We wish to do a similar thing in special Lagrangian geometry. We suppose that M is a Calabi–Yau (or more generally almost Calabi–Yau) manifold of dimension m over \mathbb{C} . We take $a \in H_m(M; \mathbb{Z})$. We denote by \mathcal{N} the set of all compact special Lagrangian submanifolds of M with homology class a . We give \mathcal{N} the C^∞ -topology. Mclean [15, Theorem 3.6] proves that \mathcal{N} is a manifold of dimension finite.

For compactifying \mathcal{N} we shall use varifolds and currents in geometric measure theory. As special Lagrangian varifolds are calibrated in the sense of Harvey and Lawson [7] they may be oriented and regarded as currents. By a fundamental theorem of Harvey and Lawson special Lagrangian currents are homologically area-minimizing, and so their area will not be cancelled; for the details we refer e.g. to Federer [5, 5.4.2] or Simon [18, Theorem 34.5]. In effect we may use varifolds in place of currents and vice versa. We shall mainly use varifolds rather than currents.

We denote by \mathcal{V} the set of all special Lagrangian integral varifolds (or currents) in M with support compact, boundary 0 and homology class a . We give \mathcal{V} the weak topology of varifolds. From an integral compactness theorem of Allard [1, Theorem 6.4] we see that \mathcal{V} is compact. We may regard \mathcal{N} as a subset

of \mathcal{V} . From a regularity theorem of Allard [1, Theorem 8.19] we see that \mathcal{N} is an open subset of \mathcal{V} , and the C^∞ -topology on \mathcal{N} is the same as the one induced from \mathcal{V} . We denote by $\overline{\mathcal{N}}$ the closure of \mathcal{N} in \mathcal{V} . We put $\partial\mathcal{N} = \overline{\mathcal{N}} \setminus \mathcal{N}$.

In the situation of Donaldson's result [4] the boundary of the moduli space consists of instantons with isolated singularities, to which one can apply the gluing method of Taubes. In our situation, however, $\partial\mathcal{N}$ consists of special Lagrangian varifolds whose singularities may be non-isolated or branched, and it will be difficult therefore to determine a neighbourhood of $\partial\mathcal{N}$ in \mathcal{V} . What we shall do is to take a certain point X of $\partial\mathcal{N}$ and determine a neighbourhood of X in \mathcal{V} .

We have mainly two things to do: the analysis of bubbling-off and the classification of local models. In the proof of Donaldson's surjectivity theorem one uses a technique of Uhlenbeck [20] for the analysis of bubbling-off, and a result of Atiyah, Hitchin and Singer [3] for the classification of local models.

We suppose that X has singularity only at one point and modelled on Jacobi-integrable smooth cones of multiplicity 1. We can then analyse the bubbling-off near X , using a technique of Allard-Almgren [2], Simon [16, 17] and the author [9]. On the other hand it seems difficult in general to classify local models for smoothing isolated conical singularities.

For stable T^2 -cones in the sense of Joyce [11, Definition 3.6], however, we can classify the local models. We can describe the stable T^2 -cones as follows. We put

$$C = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\} : |z_1| = |z_2| = |z_3|, z_1 z_2 z_3 \in (0, \infty)\}. \quad (0.1)$$

Harvey and Lawson [7, Chapter III.3.A, Theorem 3.1] prove that C is a special Lagrangian submanifold of $\mathbb{C}^3 \setminus \{0\}$. We put $\Sigma = C \cap S^5$. It is easy to see that Σ is diffeomorphic to T^2 . Joyce proves that C is a stable cone. Haskins [8] proves that every stable T^2 -cone may be expressed as $a(C)$ for some $a \in SU_3$.

We suppose now that M is of dimension 3 over \mathbb{C} . We suppose also that X has singularity only at one point and modelled on C with multiplicity 1. We prove that we have a local model L compatible with X in the situation of Joyce [14, Theorem 10.4]; L is a special Lagrangian submanifold of \mathbb{C}^3 asymptotic at infinity to C with multiplicity 1. For each $t > 0$ we can define tL by re-scaling L by t . We can find a real number $\tau > 0$ such that for each $t \in (0, \tau)$ we can glue tL to X by the method of gluing. Hence we get a smooth embedding $G : (0, \tau) \rightarrow \mathcal{N}$ with $G(t)$ tending to X as $t \rightarrow +0$.

We shall parametrize $G : (0, \tau) \rightarrow \mathcal{N}$ by a family of special Lagrangian 3-folds with isolated conical singularities modelled on C . We denote by \mathcal{X} the set of all elements of \mathcal{V} with supported connected, singularity only at one point and modelled on C with multiplicity 1. Joyce [11, Definition 5.6] defines a topology on \mathcal{X} . We prove that it is actually the same as the one induced from \mathcal{V} . By a result of Joyce [11, Corollary 6.11] we can take a neighbourhood \mathcal{Y} of X in \mathcal{X} which has a structure of manifold. Joyce proves [14, Proposition 10.3] also $\dim \mathcal{Y} = \dim \mathcal{N} - 1$. Making $\tau > 0$ and \mathcal{Y} smaller if necessary we can construct a one-to-one local diffeomorphism $G : (0, \tau) \times \mathcal{Y} \rightarrow \mathcal{N}$ with $G(t, Y)$ tending to Y as $t \rightarrow +0$ for each $Y \in \mathcal{Y}$.

We can extend G continuously to $[0, \tau) \times \mathcal{Y}$ by setting $G(0, Y) = Y$ for each $Y \in \mathcal{Y}$. We note that $G : [0, \tau) \times \mathcal{Y} \rightarrow \overline{\mathcal{N}}$ is a one-to-one open mapping. Our surjectivity result is:

Theorem 1. *G maps $[0, \tau) \times \mathcal{Y}$ onto a neighbourhood of X in \mathcal{V} .*

References

- [1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.
- [2] W. K. Allard and F. J. Almgren, *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. (2) **113** (1981), 215–265.
- [3] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.
- [4] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.
- [5] H. Federer, *Geometric measure theory*, Springer, 1969.
- [6] D. S. Freed and K. K. Uhlenbeck, *Instantons and four-manifolds*, Springer, 1984.
- [7] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
- [8] M. Haskins, *The geometric complexity of special Lagrangian T^2 -cones*, Invent. Math. **157** (2004), 11–70.
- [9] Y. Imagi, *A uniqueness theorem for gluing special Lagrangian submanifolds in Calabi–Yau manifolds*, preprint, arXiv:11052628.
- [10] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities I. Regularity*, Ann. Global Anal. Geom. **25** (2003), 201–258.
- [11] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities II. Moduli spaces*, Ann. Global Anal. Geom. **25** (2003), 301–352.
- [12] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities III. Desingularization, the unobstructed case*, Annal. Global Anal. Geom. **26** (2004), 1–58.
- [13] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities IV. Desingularization, obstructions and families*, Annal. Global Anal. Geom. **26** (2004), 117–174.

- [14] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities V. Survey and applications*, J. Differential Geom. **63** (2003), 279–347.
- [15] R. C. Mclean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), 705–747.
- [16] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), 525–571.
- [17] L. Simon, *Isolated singularities of extrema of geometric variational problems* in *Harmonic mappings and minimal immersions*, Springer, 1985.
- [18] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, 1983.
- [19] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geom. **17** (1982), 139–170.
- [20] K. K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Commun. Math. Phys. **83** (1982), 11–29.